Accepted Manuscript

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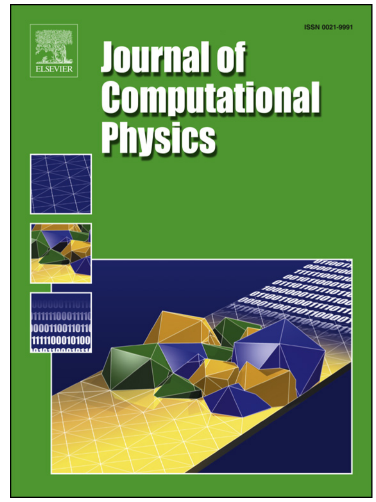

PII: S0021-9991(19)30191-3
DOI: https://doi.org/10.1016/j.jcp.2019.03.008
Reference: YJCPH 8559

To appear in: *Journal of Computational Physics*

Received date: 20 December 2018
Revised date: 1 March 2019
Accepted date: 10 March 2019

Highlights

- List of the proposals for non-integer order operators and derivatives.
- Classification of the present-day formulations of non-integer order operators.
- Discussion of criteria for classifying fractional operators.

A REVIEW OF DEFINITIONS OF FRACTIONAL DERIVATIVES AND OTHER OPERATORS

G. Sales Teodoro[a], J. A. Tenreiro Machado[b], and E. Capelas de Oliveira[c]

Abstract

Given the increasing number of proposals and definitions of operators in the scope of fractional calculus, it is important to introduce a systematic classification. Nonetheless, many of the definitions that emerged in the literature can not be considered as fractional derivatives. We analyze a list of expressions to have a general overview of the concept of fractional (integrals) derivatives. Moreover, some formulae that do not involve the term fractional, are also included due to their particular interest in the area.

1 Introduction

The non-integer order calculus, usually known as the fractional calculus (FC), began in 1695 [26, 77, 78]. The term FC is related to the letters between Leibniz and Bernoulli about the meaning of derivative of order 1/2 of the power function. Leibniz, in a brillant note, not to say prophetic, presented the correct result and stated that "...the paradox would one day have several important consequences". Today, after more than three hundred years, we testimony that the FC became a source of not only of scientific discussion and progress, but also some controversy under the light of recent proposals [13].

Several articles describe the progress in the area of FC. We can mention the works by Machado-Kiryakova-Mainardi [55, 56], with an historical review and some notes about the main scientists that promoted FC along the history. In Valério-Machado-Kiryakova [88] several pioneers in the applications of the FC are recalled, and in Valério et al. [89] a survey of useful formulas is provided, while in Machado-Kiryakova [85], a detailed review of the FC publications and conferences commemorates the 20 years of the journal Fractional Calculus and Applied Analysis.

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FC stands out in the modelling problems involving the concepts of non-locality and memory effect, that are not well explained by the integer-order calculus. Indeed, FC tackles the concept of derivative operator that, in the integer order calculus is a local operator, whereas in FC it has a non-local nature [57].

It was only after the first conference [76] dedicated exclusively to the FC, that a review of the state of the art was proposed. In particular a classification criterion was proposed for an operator to be considered fractional [77]. During the nineties of the twentieth century, some types of fractional derivatives appeared in the literature [47], and today we verify an increasing number of proposals for operators [19], both in the form of derivatives and integrals [15, 87].

We discuss and classify the proposed non-integer order operators. We restrict ourselves to the case where the order α is a real number and the order of the operator is fixed. For complex, variable- and distributed-order operators see [14, 49, 62, 68, 78] and [18, 53, 73, 100], respectively. Identically, for the quantum fractional derivative, defined on the basis of Grünwald-Letnikov derivatives [69], general fractional derivatives [44, 45], and the recent fractional and integer derivatives with continuoulsy distributed lag [84]. Finaly, for fractional integrals and derivatives as integral transforms involving special functions as kernel, and the general singular and non-singular kernels, see [78].

After our 2014 paper [15] several operators were proposed and new criteria of classification were developed. Presently, besides the criterion designed by Ross [77] (here denoted by C_1) for an operator to be considered a fractional derivative we can find in the literature one proposed by Ortigueira-Machado [39, 70] (denoted as C_2, see Section 3 in the follow-up). Furthermore, it was shown by Tarasov that for a fractional derivative the Leibniz rule is not valid. As a matter of fact, the Leibniz rule is valid for order 1, but does not hold for the higher order derivatives [82, 83].

Having this scenario in mind, we propose a classification of the non-integer order operators divided into four distinct classes, denominated: classical fractional derivatives; modified derivatives; local "fractional" operators and "fractional" operators with non-singular kernel. It is important to mention that we write the word fractional in quotation marks, since they can be considered not fractional, in the viewpoint of the criterion discussed in [39, 70]. These operators are very controversial and several distinguished researchers consider that they are not true fractional derivatives. Thus, in the follow-up the adopted nomenclature is: (i) classical derivatives, F_1 class, (ii) modified derivatives, F_2 class, (iii) local operators, F_3 class and (iv) operators with non-singular kernel, F_4 class.

The F_1 class includes those operators that began with the formulation proposed by Sonin [79], from the Riemann-Liouville integral. Liouville in 1832 [50], Gerasimov in 1948 [30] and Caputo in 1967 [16, 17] (also, although less well known, Dzhrbashyan-Nersesyan [27]), changed the order of the integral and derivative. Therefore, due to this modification, we can have a subdivision in this class, in two parts, namely the Riemann-Liouville and the Caputo types, meaning the derivative of an integral, or the integral of a derivative, respectively. In this first class we mention, in alphabetic order, the Caputo, Grünwald-Letnikov, Liouville, Riemann and Riemann-Liouville derivatives.

In the F_2 class, we have, in alphabetic order, the following formulations [87]: Canavati, Chen, Davidson-Essex, Erdélyi, Hadamard, Hilfer, Jumarie, Kober, Marchaud, and Weyl [15]. Among the most recent, also in alphabetic order, we mention: ψ-Caputo [4], Caputo-Hadamard [5], Caputo type fractional derivative [64], Hilfer-Katugampola [63], k-fractional Hilfer [25], (k, ρ)-fractional [65], Ortigueira [66, 67], ψ-Hilfer [90], and ψ-Riemann-Liouville [43].

In the F_3 class, we include the so-called local operators. This class of formulations emerged by the end of the nineties and for a review we suggest interested readers to see [46]. The most recent formulations are: Katugampola [40, 41]; conformable [42] leading to the so-called conformable calculus [1]; $\mathcal{M}$-operator [91] and $\mathcal{M}$- and $\mathcal{V}$-truncated operators [92, 93, 94, 95].

Finally, the F_4 class includes the operators with non-singular kernel. This class includes the Caputo-Fabrizio [19, 71], where the formulation was introduced, and the Losada-Nieto [54] that proposed the corresponding integral formulation and studied several properties. In this class we find also the most recent formulations: Atangana [8]; Atangana-Baleanu [9]; Yang et al. [98]; Agarwal et al. [2]; Garra et al. [29] and Panchal et al. [74]. Also, we cite the paper by Zhao-Luo, where a general fractional derivative with memory effect is introduced [102].

It is important to note four preliminary issues. We find several articles that confront two or more formulations [31] and discuss their validity for being considered a fractional operator [82, 83]. Second, this paper does not intends to present any criticism to a given researcher or proposal. The paper simply analyses present day known proposals that emerged in a fast moving scientific area. In fact, we believe that an open discussion under the light of the intellectual and gentlemen's behaviour has been the playground of the true Science for centuries. Therefore, any non-scientific discussions or personal disputes are outside of the classical and ethical formalism that the authors try to adopt here. Third, we apologize if it is missing some proposal or formulation relevant in the scope of this paper. Fourth, the paper does not intendes to cover details on the historical developments of FC.

The work is organized as follows. In section 2, we present the fractional operators, citing the corresponding specific reference. We discuss the local operators and we include the definitions, since they were not listed in [15]. Additionally, we cover the fractional derivatives with non-singular kernel. In section 3, a possible criterion for the formulations of the third and fourth classes are discussed, and several tables clarify their details and properties. Finally, in section 4 we summarize the concluding remarks.

2 Non-integer order derivatives

Here we present the fractional classical derivatives, modified derivatives, local operators and operators with non-singular kernel.

2.1 F_1 class: Classical derivatives

The so-called Grünwald-Letnikov derivative was introduced in a general form by Liouville [50], considered by some the father of FC. Later, Grünwald [32] and Letnikov [48], in 1867 and 1868, respectively, addressed this definition more specifically. Recently, this formula was extended to the complex plane [68].

This formulation is important in numerical problems and generalizes the ordinary differentiation, by means of a series [78].

We consider:

- Grünwald-Letnikov left-sided derivative

$$ {}^{\mathtt{GL}}\mathrm{D}^{\alpha}_{a^+}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1) f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \; nh = x-a $$

- Grünwald-Letnikov right-sided derivative

$$ {}^{\mathtt{GL}}\mathrm{D}^{\alpha}_{b^-}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1) f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \; nh = b-x $$

- Riemann-Liouville left-sided derivative

$$ {}^{\mathtt{RL}}\mathrm{D}^{\alpha}_{a^+}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi)\,\mathrm{d}\xi,\ x \geq a $$

- Riemann-Liouville right-sided derivative

$$ {}^{\mathtt{RL}}\mathrm{D}^{\alpha}_{b^-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b (\xi-x)^{n-\alpha-1} f(\xi)\,\mathrm{d}\xi,\ x \leq b $$

- Caputo left-sided derivative

$$ {}^{C}\mathrm{D}^{\alpha}_{a^+}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi,\ x \geq a $$

- Caputo right-sided derivative

$$ {}^{C}\mathrm{D}^{\alpha}_{b^-}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (\xi-x)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi,\ x \leq b $$

2.2 F_2 class: Modified derivatives

Here, we include the modified derivatives, that are obtained by means of a particular modification of one classical derivative. The Hilfer derivative [34] retrieves the Riemann-Liouville and Caputo derivatives for particular values of the parameters. Jumarie proposed the modified Riemann-Liouville operator [36, 37, 51], with the purpose of correcting the fact that the Riemann-Liouville derivative of the constant, that must not be confused with the so-called Heaviside unit step, whose derivative is not necessarily zero. Also, the Hilfer-Katugampola operator [63], depending on the appropriate choice of parameters recovers those derivatives from Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, Liouville, and Weyl. This formulation is defined in terms of generalized fractional integral, and can be viewed as a generalization of the Riemann-Liouville and Hadamard integrals.

More recently we find the formulation by Sousa and Oliveira [90] for the so-called ψ-Hilfer derivative. Sugumarana et al. [80] presented necessary conditions for the existence of solution of a differential equation with the ψ-Hilfer derivative. The ψ-Hilfer formulation admits the particular cases of Caputo [16], Weyl [96], Chen [21], Jumarie [36, 37, 51], ψ-Caputo

[4], ψ-Riemann-Liouville [43], Katugampola [40], Hadamard [33], Caputo-Hadamard [5], Caputo-Katugampola [7], Hilfer-Hadamard [38], Hilfer-Katugampola [63], Riemann [59], Prabhakar [75], Erdélyi-Kober [43], Hilfer [34], Liouville [15], Liouville-Caputo [61], Riesz [15], Feller [58, 103], and Riesz-Caputo [35].

We consider:

- Weyl

$$ {}_x\mathrm{D}_\infty^\alpha[f(x)] = \mathrm{D}_-^\alpha[f(x)] = (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n [{}_x\mathrm{W}_\infty^\alpha[f(x)]] $$

with $[{}_x\mathrm{W}_\infty^\alpha[f(x)]] = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_x^\infty (t-x)^{\alpha-1} f(t)\,\mathrm{d}t.$

- Marchaud

$$ \mathrm{D}_+^\alpha[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}} \mathrm{d}\xi $$

- Hadamard

$$ \mathrm{D}_+^\alpha[f(x)] = \frac{x}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \left(\ln\frac{x}{\tau}\right)^{2-\alpha} f(\tau)\frac{\mathrm{d}\tau}{\tau} $$

- Chen

$$ \mathrm{D}_\mathsf{c}^\alpha[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_\mathsf{c}^x (x-\xi)^{-\alpha} f(\xi)\,\mathrm{d}\xi,\ x \geq \mathsf{c} $$

- Davidson-Essex

$$ \mathrm{D}_0^\alpha[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}^{n+1-k}}{\mathrm{d}x^{n+1-k}} \int_0^x (x-\xi)^{-\alpha} \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}[f(\xi)]\,\mathrm{d}\xi $$

- Canavati

$$ {}_a\mathrm{D}_x^\nu[f(x)] = \frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-\xi)^\mu \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi,\ n = \lfloor\nu\rfloor,\ \mu = n-\nu $$

- Jumarie

$$\mathrm{D}_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x (x-\xi)^{n-\alpha-1}[f(\xi)-f(0)]\,\mathrm{d}\xi$$

- Erdélyi-Kober derivative [78]:

$$D_{a+;\sigma,\eta}^\alpha f(x) = x^{-\sigma(\alpha+\eta)}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(\alpha+n+\eta)} I_{a+;\sigma,\eta}^{\alpha+n} f(x),\ \alpha > -n,$$

with

$$I_{a+;\sigma,\eta}^\alpha f(x) \quad = \quad \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x f(\tau)(x^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma\eta+\sigma-1}\mathrm{d}\tau,\ \alpha > 0$$

and

$$D_{b^-;\sigma,\eta}^\alpha f(x) = x^{\sigma\eta}\left(-\frac{1}{\sigma x^{\sigma-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x^{\sigma(n-\eta)} I_{b^-;\sigma,\eta-n}^{\alpha+n} f(x),\ \alpha > -n,$$

with

$$I_{b^-;\sigma,\eta}^\alpha f(x) \quad = \quad \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^b f(\tau)(\tau^\sigma-x^\sigma)^{\alpha-1}\tau^{\sigma(1-\alpha-\eta)-1}\mathrm{d}\tau,\ \alpha > 0$$

- Regularized Liouville derivative [72]:

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \tau^{-\alpha-1}\left[f(t-\tau) - \sum_{m=0}^{N-1}\frac{(-1)^m f^{(m)}(t)}{m!}\tau^m\right]\mathrm{d}\tau,$$

with $N = \lfloor\alpha\rfloor + 1$ and $\lfloor\alpha\rfloor$ the integer part of α.

- Riesz/Feller derivative [72]:

$$D_\theta^\alpha f(t) = \frac{1}{2\sin(\alpha\pi)\Gamma(-\alpha)}\int_{\mathbb{R}} f(t-\tau)\sin\left[(\alpha+\theta\cdot\mathrm{sgn}(\tau))\frac{\pi}{2}\right]|\tau|^{-\alpha-1}\,\mathrm{d}\tau,$$

with $\theta \in \mathbb{R}$ and $\mathrm{sgn}(\cdot)$ denoting the signal function.

- Two-sided derivative [72]:

$$D_{\mathsf{C}}^{\gamma} f(t) = \lim_{h\to 0^+} h^{-\gamma} \sum_{n=-\infty}^{+\infty} (-1)^n \frac{\Gamma(\gamma+1)}{\Gamma\left(\frac{\gamma+\theta}{2} - n + 1\right)\Gamma\left(\frac{\gamma-\theta}{2} + n + 1\right)} f(t-nh),$$

with $\gamma > -1$.

- Hilfer derivative [34]:

$$D_{a\pm}^{\alpha,\mu} f(t) = \pm I_{a\pm}^{\mu(1-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) I_{a\pm}^{(1-\mu)(1-\alpha)} f(t),\ 0 \le \mu \le 1,$$

where $0 < \alpha < 1$ and

$$\begin{aligned} I_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau,\ t \ge a \\ I_{b-}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1} \mathrm{d}\tau,\ t \le b. \end{aligned}$$

- k-Hilfer derivative [25]:

$${}^{k}D^{\mu,\nu} f(t) = I_k^{\nu(1-\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) I_k^{(1-\mu)(1-\nu)} f(t),\ 0 \le \mu \le 1,$$

where $0 < \nu < 1$ and

$$I_k^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t f(\tau)(t-\tau)^{\frac{\alpha}{k}-1} \mathrm{d}\tau,\ t \ge 0.$$

- Hilfer-Katugampola [63]:

$${}^{\rho}D_{a\pm}^{\alpha,\beta} f(x) = \left[\pm {}^{\rho}I_{a\pm}^{\beta(1-\alpha)} \left(t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right) {}^{\rho}I_{a\pm}^{(1-\beta)(1-\alpha)}\right] f(t),\ \rho > 0,$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$ and

$$\begin{aligned} {}^{\rho}I_{a+}^{\alpha} f(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x f(\tau)(x^{\rho}-\tau^{\rho})^{\alpha-1} \mathrm{d}\tau,\ x > a \\ {}^{\rho}I_{b-}^{\alpha} f(x) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b f(\tau)(\tau^{\rho}-x^{\rho})^{\alpha-1} \mathrm{d}\tau,\ x < b. \end{aligned}$$

- ψ-Hilfer derivative [90]:

$$ {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x),\ 0 \le \beta \le 1, $$

and

$$ {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi}\left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x),\ 0 \le \beta \le 1. $$

Here $n-1<\alpha<n$ with $n \in \mathbb{N}$, $I=[a,b]$ is an interval such that $-\infty \le a < b \le \infty$, ψ denotes an increasing function such that $\psi'(x) \neq 0$ for all $x \in I$ and $\psi \in C^n([a,b],\mathbb{R})$. The corresponding integral (on the left and on the right) are given by

$$ I_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)[\psi(x)-\psi(t)]^{\alpha-1} f(t)\mathrm{d}t $$

and

$$ I_{b-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)[\psi(t)-\psi(x)]^{\alpha-1} f(t)\mathrm{d}t\,, $$

respectively.

The Hilfer, k-Hilfer and ψ-Hilfer derivatives are given in terms of two integrals and one derivative of integer order. We note that, differently from other formulations, the Hilfer-Katugampola, recovers as particular cases, the Riemann-Liouville $\rho=1$ and $\beta=0$; Caputo $\rho=1=\beta$; Hilfer $\rho=1$; Weyl $\rho=1$, $\beta=0$ and $a\to-\infty$; Hilfer-Hadamard $\rho\to0^+$; Caputo-Hadamard $\rho\to0^+$ and $\beta=1$ and Hadamard $\rho\to0^+$ and $\beta=0$. The most general is ψ-Hilfer derivative that generalizes twenty-two other formulations, mentioned above. We omit their definitions here, and for the values of the parameters that recovers each of them see [90].

2.3 F_3 class: Local operators

In this sub-section we present a brief summary of the so-called local formulations. These formulations should not contain the name fractional, since it was recently shown [83] that, at most, they are a multiplicative factor of the derivative of order one. In spite of this, such proposals that appeared in the late nineties of the twentieth century had some widespread since 2010.

Chen, in 2006, presented the local operator in order to model phenomena of turbulence [22] and anomalous diffusion [23]. The conformable operator was proposed by Khalil et al. [42] in 2014. We find its applications in Newtonian mechanics [24], diffusion equation [28] and the solution of a nonlinear differential equation [12]. In 2017, the so-called deformable operator [104] was advanced since the proposal in [42] does not include zero as a possibility for the order of the derivative. In 2014, the Katugampola formulation [40] was introduced being used in quantum mechanics. More recently, in 2018, Sousa and Oliveira [91] defined the $\mathcal{M}$-operator as a generalization of the one proposed by Katugampola [40]. The beta operator brought up by Atangana and Goufo [10] was used in problems involving asymptotic methods [10] and mathematical models describing infectious diseases [11]. In 2016, Almeida et al. [6], introduced an operator, here denoted as AGO, generalizing the beta [10] and conformable [42] formulations. In 2017, Akkurt et al. [3] proposed another generalization encompassing the Katugampola [40], AGO [6] and conformable [42] operators. Recently, in 2018, Vanterler-Oliveira [92, 94, 95] proposed three operators, namely the $\mathcal{V}$-truncated, the $\mathcal{V}$-truncated with a Mittag-Leffler function (MLF) with six parameters, and the $\mathcal{M}$-truncated, that unifies a series of expressions that support the properties of the integer-order derivative.

We now present the local operators definitions. In what follows we consider f a real function, $0 < \alpha \leq 1$, $k : [a, b] \to \mathbb{R}$ a continuous nonnegative map such that $k'(x) \neq 0$, whenever $x > a > 0$ and $E_\alpha(x)$ the classical MLF with one parameter.

We consider:

- Kolwankar [46]:

$$D^\alpha f(x) = \lim_{x' \to x} D_x^\alpha [f(x') - f(x)],$$

where D_x^α is the Riemann-Liouville fractional derivative.

- Chen [23]:

$$\frac{\partial f(x)}{\partial x^\alpha} = \lim_{s \to x} \frac{f(x) - f(s)}{x^\alpha - s^\alpha}.$$

- Conformable [42]:

$$T_\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

- Katugampola [40]:

$$D^{\alpha}f(x) = \lim_{\varepsilon\to 0}\frac{f(xe^{\varepsilon x^{-\alpha}}) - f(x)}{\varepsilon}.$$

- $\mathcal{M}$ [91]:

$$\mathscr{D}_M^{\alpha,\beta}f(x) = \lim_{\varepsilon\to 0}\frac{f(xE_{\beta}(\varepsilon x^{-\alpha})) - f(x)}{\varepsilon},\ \beta > 0.$$

- Deformable [104]:

$$\mathcal{D}^{\alpha}f(x) = \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)f(x+\varepsilon\alpha) - f(x)}{\varepsilon},\ \alpha+\beta = 1.$$

- Beta [10]:

$${}_0^A D_x^{\beta}(f(x)) = \lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\varepsilon},\ \beta\in(0,1].$$

- AGO [6]:

$$f^{(\alpha)}(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon(k(x))^{1-\alpha}) - f(x)}{\varepsilon}.$$

- Generalized [3]:

$${}_G D^{\alpha}f(x) = \lim_{\varepsilon\to 0}\frac{f\left(x-k(x)+k(x)e^{\frac{\varepsilon(k(x))^{-\alpha}}{k'(x)}}\right) - f(x)}{\varepsilon}.$$

- $\mathcal{V}$-truncated [94]:

$${}_i^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f(x) = \lim_{\varepsilon\to 0}\frac{f\left(x\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}(\varepsilon x^{-\alpha})\right) - f(x)}{\varepsilon},$$

where $\gamma,\beta,\rho,\delta\in\mathbb{C}$, $p,q>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\gamma)+p\geq q$ and

$${}_iH_{\gamma,\beta,p}^{\rho,\delta,q}(x) = \Gamma(\beta)\sum_{k=0}^{i}\frac{x^k(\rho)_{kq}}{\Gamma(\gamma k+\beta)(\delta)_{kp}}.$$

- Conformable of β-type in the Riemann-Liouville sense [60]:

$$^{AR}\,{}^{\beta}_{a}\mathcal{D}^{\alpha}_{t}f(x) = {}^{A}\,{}^{n}_{a}\mathcal{D}^{\alpha}_{t}({}^{A}\,{}^{n-\beta}_{a}\mathcal{I}_{t}f(x)), \qquad \mathrm{Re}(\beta) > 0,\ n = [\mathrm{Re}(\beta)] + 1.$$

- General conformable [101]:

$$D^{p}_{\psi}f(u) = \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon\psi(u,p)) - f(u)}{\varepsilon}.$$

2.4 F_4 class: Operators with non-singular kernel

In this sub-section, we present briefly the so-called operators with non-singular kernel. Many of these formulations involve the name fractional, but it was shown that none of them satisfies the criterion C_2 (see section 3), to be considered a fractional derivative. In 2016 Yang et al. [98] proposed a new formulation involving an integral whose kernel is an exponential function. Atangana-Baleanu [9], presented two expressions with a kernel based on the MLF. Teodoro-Oliveira [86] formulated a "general" operator that contains as particular cases the Caputo-Fabrizio [19], Yang et al. [98], and Atangana-Baleanu [9] operators. In 2016, Caputo-Fabrizio [20] introduced a new formula with a Gasussian kernel. In 2017, Sun et al. [81] proposed two new formulations with the kernel composed by an exponential function. Still in 2017, Yang et al. [99] proposed the Riemann-Liouville and Liouville-Caputo type expessions with kernels involving MLF with one, two, three and four parameters. Finally, in 2018, Yang [97] brought out two formulations, the general Liouville-Caputo and Riemann-Liouville types.

Below, we present the definitions of operators with non-singular kernel. In what follows we consider $E_{\alpha}(t)$, the classical MLF with one parameter α, and $E^{\rho,q}_{\alpha,\beta}(t)$, the MLF with four parameters (α, β, ρ, q). Let us also consider $0 < \alpha < 1$ and $f \in H^1(a,b)$ where $b > a$. The functions (as defined in original papers) $M(\alpha)$, $B(\alpha)$, $R(\alpha)$ and $G(\alpha)$ denote normalization functions obeying $M(0) = 1 = M(1)$, $B(0) = 1 = B(1)$, $R(0) = 1 = R(1)$ and $G(0) = 1 = G(1)$.

- Caputo-Fabrizio [19]:

$$\mathscr{D}^{(\alpha)}_{t}f(t) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{t}\dot{f}(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\,\mathrm{d}\tau.$$

- Atangana-Baleanu Caputo type [9]:

$$ {}^{ABC}{}_{b}D_{t}^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_{b}^{t} f'(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x,\ t>b. $$

- Atangana-Baleanu Riemann-Liouville type [9]:

$$ {}^{ABR}{}_{b}D_{t}^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{b}^{t} f(x)E_{\alpha}\left(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right)\mathrm{d}x,\ t>b. $$

- Yang et al. [98]:

$$ {}^{YSM}D_{a+}^{(\alpha)}f(t) = \frac{R(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t} f(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau,\ t>a. $$

- Generalized Caputo type [86]:

$$ {}^{gC}D_{t}^{\alpha,\beta}(f(t)) = \frac{G(\alpha)}{1-\alpha}\int_{b}^{t} f'(x)E_{\beta}\left(-\alpha\frac{(t-x)^{\beta}}{1-\alpha}\right)\mathrm{d}x,\ \beta\in[0,1],\ t>b. $$

- Generalized Riemann-Liouville type [86]:

$$ {}^{gRL}D_{t}^{\alpha,\beta}(f(t)) = \frac{G(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{b}^{t} f(x)E_{\beta}\left(-\alpha\frac{(t-x)^{\beta}}{1-\alpha}\right)\mathrm{d}x,\ \beta\in[0,1],\ t>b. $$

- Caputo-Fabrizio with Gaussian kernel [20]:

$$ {}^{CF}D^{\alpha}f(t) = \frac{1+\alpha^{2}}{\sqrt{\pi^{\alpha}(1-\alpha)}}\int_{a}^{t}\dot{f}(\tau)e^{-\frac{\alpha(t-\tau)^{2}}{1-\alpha}}\mathrm{d}\tau,\ f(a)=0,\ t>a. $$

- Sun-Hao-Zhang-Baleanu [81]:

$$ {}^{SE}D_{a+}^{\alpha}f(t) = \frac{M(\alpha)}{(1-\alpha)^{\frac{1}{\alpha}}}\int_{a}^{t} f'(\tau)e^{-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}}\mathrm{d}\tau,\ t>a. $$

- Yang et al. Liouville-Caputo type [99]:

$$ {}^{C}_{E_{\varphi,\phi}}D_{\alpha}^{(\nu)}f(t) = \int_{a}^{t} E_{\varphi,\phi}((\nu_{1},v_{1}),\cdots,(\nu_{n}v_{n});(t-\tau)^{\nu})\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau)\mathrm{d}\tau, $$

- Yang et al. Riemann-Liouville type [99]:

$$ {}^{RL}_{E_{\varphi,\phi}}D_{\alpha}^{(\nu)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t E_{\varphi,\phi}((\nu_1, v_1), \cdots, (\nu_n v_n); (t-\tau)^{\nu}) f(\tau) \mathrm{d}\tau, $$

- General Riemann-Liouville type [97]:

$$ \mathbb{D}_{(\Xi)}^{\mathrm{RL}} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \Xi(t-\tau) f(\tau) \mathrm{d}\tau, \ t > 0. $$

- General Liouville-Caputo type [97]:

$$ \mathbb{D}_{(\Xi)}^{\mathrm{C}} f(t) = \int_0^t \Xi(t-\tau) f'(\tau) \mathrm{d}\tau, \ t > 0. $$

After presenting the four classes (F_1, F_2, F_3, F_4), we discuss possible criteria for analysing the operators.

3 Criteria and tables

In 1975 Ross [77] proposed a criterion, C_1, that an operator must satisfy to be considered a fractional derivative. The criterion C_1 includes the following conditions:

R_1^1 : the fractional derivative of an analytic function is also analytic;

R_1^2 : (i) when the order is a positive integer, the fractional derivative, must produce the same result of the ordinary derivative, (ii) when the order is a negative integer the fractional derivative must produce the same result of the n-th ordinary integration;

R_1^3 : the zero order derivative of a function is the function itself;

R_1^4 : the fractional derivative is a linear operator;

R_1^5 : the semigroup property is satisfied.

The criterion C_2, was proposed in 2015 by Ortigueira et al. [70] and includes the following conditions:

R_2^1 : the fractional derivative is a linear operator;

R_2^2 : the zero order derivative of a function is the function itself;

R_2^3 : when the order is a positive integer, the fractional derivative, must produce the same result of the ordinary derivative;

R_2^4 : the $D^\alpha D^\beta f(x) = D^{\alpha+\beta} f(x)$ exponent law is satisfied for $\alpha < 0$ and $\beta < 0$;

R_2^5 : the Generalized Leibniz Type Rule is valid

$$D^\alpha[f(x)g(x)] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)k!} D^k f(x) D^{\alpha-k} g(x)\cdot$$

The two criteria C_1 and C_2 include five properties, that differ in only one property: in C_1 we have R_1^1 "The fractional derivative of an analytic function is analytic", while in C_2 we find the property R_2^5 "Generalized Leibniz Type Rule".

Another way to classify the operators, here denoted by C_3, is due to Tarasov [82, 83], that justifies a fractional derivative through nonlocality and violation of the Leibniz rule. Linear operators satisfying the classical Leibniz rule are not fractional derivatives [82], since for this rule to be satisfied, the order of the fractional derivative must coincide with the differentiation order of one.

We present in the follow-up Table 1 with the corresponding generalized Leibniz rule for some operators of the F_1 class: classical derivatives and F_2 class: modified derivatives.

Table 1: Generalized Leibniz type rule for classical and modified derivatives.

Classical derivative	Generalized Leibniz rule
Grünwald-Letnikov	$_{GL}D^{\alpha}(fg)(x) = \sum_{i=0}^{\infty} \binom{\alpha}{i} f^{(i)}(x) {}_{GL}D^{\alpha-i}g(x)$
Riemann-Liouville	$D^{\alpha}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(x) D^{\alpha-k}g(x)$
Caputo	${}^{C}D^{\alpha}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_{*}D^{k}f(x) {}_{*}D^{\alpha-k}g(x) + g(0)(f(x) - f(0)) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$
Hilfer	$D_{a\pm}^{\alpha,\mu}(fg)(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} f^{(m)}(x) D_{a\pm}^{\alpha-m,\mu}g(x)$ $+ \sum_{k=0}^{\infty} \binom{-(1-\mu)(1-\alpha)}{k} I_{a\pm}^{k+(1-\mu)(1-\alpha)} g(a)(f^{(k)}(x) - f^{(k)}(a)) \frac{(x-a)^{-\alpha\mu+\mu-1}}{\Gamma(\mu-\alpha\mu)}$
Weyl	$W_{\pm}^{\alpha}(fg)(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} f^{(m)}(x) W_{\pm}^{\alpha-m}g(x)$
ψ-Hilfer	${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}(fg)(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{-(1-\beta)(1-\alpha)}{m-l} \binom{\beta(\alpha-1)+1}{l} f^{(m)}(x) {}^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x)$ $- \sum_{k=0}^{\infty} \binom{-(1-\beta)(1-\alpha)}{k} I_{a+}^{(1-\beta)(1-\alpha)+k;\psi} g(a) f^{(k)}(a) \frac{[\psi(x)-\psi(a)]^{-1-\beta(\alpha-1)}}{\Gamma(\beta(1-\alpha))}$
Hadamard	$\mathtt{D}_{a+}^{\alpha} fg(x) = \sum_{m=0}^{\infty} \left[\binom{\alpha-1}{m} + \binom{\alpha-1}{m-1} \right] f^{(m)}(x) {}^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x).$
Erdélyi-Kober	$D_{a+;\sigma,\eta}^{\alpha} fg(x) = x^{-\sigma(2\eta+\alpha)} \sum_{m=0}^{\infty} \left[\binom{\alpha-1}{m} + \binom{\alpha-1}{m-1} \right] \sum_{k=0}^{m} \frac{\Gamma(-\sigma(\eta+\alpha)+1)}{\Gamma(-\sigma(\eta+\alpha)-k+1)} x^{-k} f^{(m-k)}(x) {}^{RL}\mathbb{D}_{a+}^{\alpha-m;\psi}g(x)$
ψ-Caputo	${}^{C}\mathbb{D}_{a+}^{\alpha;\psi}(fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(x) {}^{RL}\mathbb{D}_{a+}^{\alpha-k;\psi}g(x) - \sum_{k=0}^{n-1} \frac{\frac{d^k}{dx^k}[f(x)g(x)](a)}{\Gamma(k-\alpha+1)} [\psi(x)-\psi(a)]^{k-\alpha}$

Local operators, with exception of deformable case, satisfy the classical Leibniz rule as shown in Table 2. Nevertheless, this is due to the relationship of these operators with the derivative of order one.

Table 2: Applications of the C_2 criteria to the local operators.

Local operator	Linearity	Zero order	Semigroup	Integer order	Generalized Leibniz rule
Chen	✓	×	×	⋄	×
Conformable	✓	×	×	✓	×
Katugampola	✓	×	×	✓	×
$\mathcal{M}$	✓	×	×	✓	×
Deformable	✓	✓	×	✓	×
Beta	✓	×	×	✓	×
AGO	✓	×	×	✓	×
Generalized	✓	⋄	×	⋄	×
$\mathcal{V}$-trucated	✓	×	×	⋄	×

Note: the symbol ✓represents that the property is satisfied, × that the property is not satisfied and ⋄ that it is satisfied for special cases only.

Considering that the local formulations do not satisfy the criteria C_2 and C_3, here we investigate one possible alternative criteria, denoted by C_4, composed of five conditions, as follows:

R_4^1 : the local derivative is a linear operator;

R_4^2 : the local derivative of order one must produce the same result of the first ordinary derivation;

R_4^3 : the local derivative of a constant is zero;

R_4^4 : the classic Leibniz rule holds

$$D^\alpha[fg](t) = g(t)D^\alpha[f(t)] + f(t)D^\alpha[g(t)];$$

R_4^5 : the chain rule holds

$$D^\alpha[f(g(t)] = D^\alpha f(t)(g(t))g'(t)\cdot$$

Table 3 shows the properties for the local operators in the point of view of criterion C_4.

Table 3: Applications of the criterion C_4 to the local operators.

Local operator	Linearity	One order	Derivative of constant	Leibniz rule	Chain rule
Chen	✓	✓	✓	✓	✓
Conformable	✓	✓	✓	✓	✓
Katugampola	✓	✓	✓	✓	✓
$\mathcal{M}$	✓	✓	✓	✓	✓
Deformable	✓	✓	$\diamond$	$\diamond$	$\diamond$
Beta	✓	✓	✓	✓	✓
AGO	✓	✓	✓	✓	✓
Generalized	✓	$\diamond$	✓	✓	✓
$\mathcal{V}$-truncated	✓	$\diamond$	✓	✓	✓

Note: the symbol ✓represent that the property is satisfied and $\diamond$ that it is satisfied for special cases, only.

In Table 4 we depict the properties of the operators with non-singular kernel in the perspective of C_2 showing explicitly an expression for the generalized Leibniz type rule. We observe that such operators do not fulfill all the properties of the criterion C_2 and can not be called fractional derivatives.

Table 4: Properties of the C_2 criterion for non-singular kernel operators.

Non-singular kernel operator	1	2	3	4	Generalized Leibniz type rule
Caputo-Fabrizio [19]	✓	◇	◇	◇	$\mathscr{D}_t^{(\alpha)}[fg](t) = \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k \sum_{s=0}^{\infty}\binom{-k}{s} f^{(s)}(t) I_{b+}^{k+s} g(t) - \frac{M(\alpha)}{1-\alpha} e^{-\alpha\frac{(t-b)}{1-\alpha}} f(b)g(b)$
Atangana-Baleanu Caputo type [9]	✓	◇	×	✓	${}^{ABC}{}_bD_t^{\alpha}[fg](t) = \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k\sum_{s=0}^{\infty}\binom{-\alpha k}{s} f^{(s)}(t) I_{b+}^{\alpha k+s}g(t) - \frac{B(\alpha)}{1-\alpha}E_{\alpha}\left(-\alpha\frac{(t-b)^{\alpha}}{1-\alpha}\right)f(b)g(b)$
Atangana-Baleanu Riemann-Liouville type [9]	✓	✓	×	✓	${}^{ABR}{}_bD_t^{\alpha}[fg](t) = \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k\sum_{s=0}^{\infty}\binom{-\alpha k}{s} f^{(s)}(t) I_{b+}^{\alpha k+s}g(t)$
Yang et al. [98]	✓	✓	-	✓	${}^{YSM}D_{a+}^{(\alpha)}[fg](t) = \frac{R(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k\sum_{s=0}^{\infty}\binom{-k}{s} f^{(s)}(t) I_{b+}^{k+s}g(t)$
Generalized Caputo type[86]	✓	◇	-	◇	${}^{gC}D_t^{\alpha,\beta}[fg](t) = \frac{G(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k\sum_{s=0}^{\infty}\binom{-\beta k}{s} f^{(s)}(t) I_{b+}^{\beta k+s}g(t) - \frac{G(\alpha)}{1-\alpha}E_{\beta}\left(-\alpha\frac{(t-b)^{\beta}}{1-\alpha}\right)f(b)g(b)$
Generalized Riemann-Liouville type [86]	✓	✓	-	◇	${}^{gRL}D_t^{\alpha,\beta}[fg](t) = \frac{G(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^k\sum_{s=0}^{\infty}\binom{-\beta k}{s} f^{(s)}(t) I_{b+}^{\beta k+s}g(t)$
Gaussian kernel [20]	✓	✓	-	✓	${}^{CF}D^{\alpha}[fg](t) = \frac{1+\alpha^2}{\sqrt{\pi^{\alpha}(1-\alpha)}}\sum_{k=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^k\frac{(2k)!}{k!}\sum_{s=0}^{\infty}\binom{-2k}{s} f^{(s)}(t) I_{a+}^{2k+s}g(t)$
Sun-Hao-Zhang-Baleanu [81]	✓	×	-	✓	${}^{SE}D_{a+}^{\alpha}[fg](t) = \frac{M(\alpha)}{(1-\alpha)^{\frac{1}{\alpha}}}\left[e^{-\frac{\alpha(t-a)^{\alpha}}{1-\alpha}} - \sum_{k=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^k\frac{\alpha\Gamma(\alpha)}{(k-1)!}\sum_{s=0}^{\infty}\binom{-2\alpha}{s} f^{(s)}(t) I_{a+}^{2\alpha+s}g(t)\right]$
Yang et al. Caputo type[99]	✓	×	×	×	${}^{C}_{E^{\rho,q}_{\alpha,\beta}}D_a^{(\alpha)}[fg](t) = \sum_{k=0}^{\infty}\frac{(\rho)_{qk}}{\Gamma(\alpha k+\beta)}\frac{1}{k!}\Gamma(\alpha k+1)\sum_{s=0}^{\infty}\binom{-\alpha k}{s} f^{(s)}(t) I_{a+}^{\alpha k+s}g(t) - E^{\rho,q}_{\alpha,\beta}((t-a)^{\alpha})f(a)g(a)$
Yang et al. Riemann-Liouville type [99]	✓	×	×	×	${}^{RL}_{E^{\rho,q}_{\alpha,\beta}}D_a^{(\alpha)}[fg](t) = \sum_{k=0}^{\infty}\frac{(\rho)_{qk}}{\Gamma(\alpha k+\beta)}\frac{1}{k!}\Gamma(\alpha k+1)\sum_{s=0}^{\infty}\binom{-\alpha k}{s} f^{(s)}(t) I_{a+}^{\alpha k+s}g(t).$
General Liouville-Caputo type [97]	✓	◇	✓	×	${}^{CT}_{0}\mathbb{D}_t^{\alpha}[fg](t) = \sum_{k=0}^{\infty}\binom{-\alpha}{k} f^{(k)}(t)\left[{}^{CT}_{0}\mathbb{D}_t^{\alpha+k}g(t) + \frac{1}{\Gamma(\alpha+k+1)}g(0)t^{\alpha+k}\right] - \frac{1}{\Gamma(\alpha+1)}f(0)g(0)t^{\alpha}$
General Riemann-Liouville type [97]	✓	✓	✓	×	${}^{RLT}_{0}\mathbb{D}_t^{\alpha}[fg](t) = \sum_{k=0}^{\infty}\binom{-\alpha}{k} f^{(k)}(t)\,{}^{RLT}_{0}\mathbb{D}_t^{\alpha+k}g(t)$

Note: The symbol ✓represents that the property is satisfied, × that the property is not satisfied and ◇ that it is satisfied for special cases only. The columns headed with symbols 1 to 4 represent the linear operator property, order zero, semigroup and derivative of integer order, respectively [86].

4 Concluding remarks

In this paper we presented a review of present-day proposals for operators in the scope of FC. When checking the class F_1 of classical derivatives we observe that they fulfill the properties of criterion C_2. Therefore, according to C_2, the operators present in this class can be called fractional.

When analyzing the class F_3 of local operators, we verify that they can be written in terms of the ordinary derivative of order one. These operators do not meet all the conditions of the criterion C_2. With respect to the derivative of the two-function product, such operators satisfy the classical Leibniz rule, due to their relationship with the standard first-order derivative. Therefore, this class can not be considered fractional derivative, and the same conclusion is valid for F_4. The paper presented properties for C_2 for each class of formulations, with an extra focus in the generalized Leibniz type rule.

Following these ideas, we present five properties in criterion C_4 that this class of operators should satisfy. Nonetheless, the only property of the criterion that all formulations of class F_4 with non-singular kernel fulfill is that they are linear operators. For the other properties these proposals do not follow a global pattern, and the results vary from case to case.

This study can be complemented with four extra classes of derivatives, namely F_5 "Developed FD": the Integration and Differentiation of Distributed order; F_6 "Generalized FD": the FD and FI from the Generalized Fractional Calculus; F_7 Operators with "Probabilistic kernel" (Probability density function, pdf-kernel); F_8 "Hypersingular kernels" among others. They are not included here for the sake of parsimony, but it is planed for future work.

We conclude this work recalling again Leibniz about Science and Mathematics "The means of obtaining as much variety as possible, but with the greatest possible order... is the means of obtaining as much perfection as possible."

Acknowledgment

We would like to thank the referees for their valuable and detailed comments that improved this manuscript.

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