

Introduction and Implementation for Finite Element Methods

Chapter 10: Finite elements for 2D unsteady Navier-Stokes equations

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Outline

- 1 Weak formulation
- 2 Semi-discretization
- 3 Full discretization
- 4 Newton's iteration
- 5 Matrix formulation
- 6 FE method
- 7 More Discussion

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- 1 Weak formulation
- 2 Semi-discretization
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Target problem

- Consider the 2D unsteady Navier-Stokes equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f}, \quad \text{in } \Omega \times [0, T],$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T],$$

$$\mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega \times [0, T],$$

$$\mathbf{u} = \mathbf{u}_0, \quad p = p_0, \quad \text{at } t = 0 \text{ and in } \Omega.$$

where Ω is a 2D domain, $[0, T]$ is the time interval, $\mathbf{f}(x, y, t)$ is a given function on $\Omega \times [0, T]$, $\mathbf{g}(x, y, t)$ is a given function on $\partial\Omega \times [0, T]$, $\mathbf{u}_0(x, y)$ and $p_0(x, y)$ are given functions on Ω at $t = 0$, $\mathbf{u}(x, y, t)$ and $p(x, y, t)$ are the unknown functions, and

$$\mathbf{u}(x, y, t) = (u_1, u_2)^t, \quad \mathbf{f}(x, y, t) = (f_1, f_2)^t,$$

$$\mathbf{g}(x, y, t) = (g_1, g_2)^t, \quad \mathbf{u}_0(x, y) = (u_{10}, u_{20})^t.$$

Target problem

- The nonlinear advection is defined as

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- The stress tensor $\mathbb{T}(\mathbf{u}, p)$ is defined as

$$\mathbb{T}(\mathbf{u}, p) = 2\nu \mathbb{D}(\mathbf{u}) - p \mathbb{I}$$

where ν is the viscosity and the deformation tensor

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t).$$

Target problem

- In more details, the deformation tensor can be written as

$$\mathbb{D}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- Hence the stress tensor can be written as

$$\mathbb{T}(\mathbf{u}, p) = \begin{pmatrix} 2\nu \frac{\partial u_1}{\partial x} - p & \nu \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\ \nu \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & 2\nu \frac{\partial u_2}{\partial y} - p \end{pmatrix}.$$

Weak formulation

- First, take the inner product with a vector function $\mathbf{v}(x, y) = (v_1, v_2)^t$ on both sides of the unsteady Navier-Stokes equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega$$

$$\Rightarrow \mathbf{u}_t \cdot \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v} \quad \text{in } \Omega$$

$$\Rightarrow \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy$$

$$- \int_{\Omega} (\nabla \cdot \mathbb{T}(\mathbf{u}, p)) \cdot \mathbf{v} \, dx dy = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy$$

Weak formulation

- Second, multiply the divergence free equation by a function $q(x, y)$:

$$\begin{aligned}\nabla \cdot \mathbf{u} = 0 &\Rightarrow (\nabla \cdot \mathbf{u})q = 0 \\ &\Rightarrow \int_{\Omega} (\nabla \cdot \mathbf{u})q \, dx dy = 0.\end{aligned}$$

- $\mathbf{u}(x, y, t)$ and $p(x, y, t)$ are called trial functions and $\mathbf{v}(x, y)$ and $q(x, y)$ are called test functions.

Weak formulation

- Using integration by parts in multi-dimension:

$$\int_{\Omega} (\nabla \cdot \mathbb{T}) \cdot \mathbf{v} \, dx dy = \int_{\partial\Omega} (\mathbb{T}\mathbf{n}) \cdot \mathbf{v} \, ds - \int_{\Omega} \mathbb{T} : \nabla \mathbf{v} \, dx dy,$$

where $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of $\partial\Omega$, we obtain

$$\int_{\Omega} \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} \, dx dy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy.$$

Here,

$$\begin{aligned} A : B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}. \end{aligned}$$

Weak formulation

- Using the above definition for $A : B$, it is not difficult to verify (an independent study project topic) that

$$\begin{aligned} \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} &= (2\nu \mathbb{D}(\mathbf{u}) - p\mathbb{I}) : \nabla \mathbf{v} \\ &= 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - p(\nabla \cdot \mathbf{v}). \end{aligned}$$

- Hence we obtain

$$\begin{aligned} &\int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\ &- \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy, \\ &- \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0. \end{aligned}$$

Here we multiply the second equation by -1 in order to keep the matrix formulation symmetric later.

Weak formulation

- Since the solution on the domain boundary $\partial\Omega$ are given by $\mathbf{u}(x, y, t) = \mathbf{g}(x, y, t)$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega$.
- Hence

$$\begin{aligned} & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\ & - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0. \end{aligned}$$

- Define $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$ and

$$\begin{aligned} H^1(0, T; [H^1(\Omega)]^2) &= \{ \mathbf{v}(\cdot, t), \frac{\partial \mathbf{v}}{\partial t}(\cdot, t) \in [H^1(\Omega)]^2, \forall t \in [0, T] \}, \\ L^2(0, T; L^2(\Omega)) &= \{ q(\cdot, t) \in L^2(\Omega), \forall t \in [0, T] \}. \end{aligned}$$

Weak formulation

- Weak formulation in the vector format: find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned} & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy \\ & + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy \\ & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0, \end{aligned}$$

for any $\mathbf{v} \in [H_0^1(\Omega)]^2$ and $q \in L^2(\Omega)$.

Weak formulation

- Define

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy,$$

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy,$$

$$b(\mathbf{u}, q) = - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy,$$

$$(\mathbf{f}, \mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy.$$

- Weak formulation: find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned} (\mathbf{u}_t, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}), \\ b(\mathbf{u}, q) &= 0, \end{aligned}$$

for any $\mathbf{v} \in [H_0^1(\Omega)]^2$ and $q \in L^2(\Omega)$.

Weak formulation

- In more details,

$$\begin{aligned}
 & \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
 = & \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & \frac{\partial u_2}{\partial y} \end{pmatrix} \\
 & : \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{1}{2} \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) & \frac{\partial v_2}{\partial y} \end{pmatrix} \\
 = & \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + \frac{1}{4} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \\
 & + \frac{1}{4} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) + \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y}.
 \end{aligned}$$

Weak formulation

- Hence

$$\begin{aligned}
 & \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
 = & \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + \frac{1}{2} \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} \\
 & + \frac{1}{2} \frac{\partial u_1}{\partial y} \frac{\partial v_2}{\partial x} + \frac{1}{2} \frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{1}{2} \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x}.
 \end{aligned}$$

- Then

$$\begin{aligned}
 & \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\
 = & \int_{\Omega} \nu \left(2 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + 2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} \right. \\
 & \left. + \frac{\partial u_1}{\partial y} \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} \right) dx dy.
 \end{aligned}$$

Weak formulation

- We also have

$$\begin{aligned}
 \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy &= \int_{\Omega} \frac{\partial u_1}{\partial t} v_1 \, dx dy + \int_{\Omega} \frac{\partial u_2}{\partial t} v_2 \, dx dy, \\
 \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy \\
 &= \int_{\Omega} \left(u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right) dx dy, \\
 \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy &= \int_{\Omega} \left(p \frac{\partial v_1}{\partial x} + p \frac{\partial v_2}{\partial y} \right) dx dy, \\
 \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy &= \int_{\Omega} (f_1 v_1 + f_2 v_2) \, dx dy, \\
 \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy &= \int_{\Omega} \left(\frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right) dx dy.
 \end{aligned}$$

Weak formulation

- Weak formulation in the scalar format: find $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$, and $p \in L^2(\Omega)$ such that

$$\begin{aligned}
 & \int_{\Omega} \frac{\partial u_1}{\partial t} v_1 \, dx dy + \int_{\Omega} \frac{\partial u_2}{\partial t} v_2 \, dx dy \\
 & + \int_{\Omega} \left(u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right) \, dx dy \\
 & + \int_{\Omega} \nu \left(2 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + 2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} \right. \\
 & \left. + \frac{\partial u_1}{\partial y} \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} \right) \, dx dy \\
 & - \int_{\Omega} \left(p \frac{\partial v_1}{\partial x} + p \frac{\partial v_2}{\partial y} \right) \, dx dy = \int_{\Omega} (f_1 v_1 + f_2 v_2) \, dx dy. \\
 & - \int_{\Omega} \left(\frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right) \, dx dy = 0.
 \end{aligned}$$

for any $v_1 \in H_0^1(\Omega)$, $v_2 \in H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

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Galerkin formulation

- Consider a finite element space $U_h \subset H^1(\Omega)$ for the velocity and a finite element space $W_h \subset L^2(\Omega)$ for the pressure. Define U_{h0} to be the space which consists of the functions of U_h with value 0 on the Dirichlet boundary.
- Then the Galerkin formulation is to find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$\begin{aligned} (\mathbf{u}_{h_t}, \mathbf{v}) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in [U_{h0}]^2$ and $q_h \in W_h$.

Galerkin formulation

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$\begin{aligned} (\mathbf{u}_{h_t}, \mathbf{v}) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

Galerkin formulation

- In more details of the vector format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$\begin{aligned}
 & \int_{\Omega} \mathbf{u}_{ht} \cdot \mathbf{v}_h \, dx dy + \int_{\Omega} (\mathbf{u}_h \cdot \nabla) \mathbf{u}_h \cdot \mathbf{v}_h \, dx dy \\
 & + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \, dx dy - \int_{\Omega} p_h (\nabla \cdot \mathbf{v}_h) \, dx dy \\
 & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h \, dx dy, \\
 & - \int_{\Omega} (\nabla \cdot \mathbf{u}_h) q_h \, dx dy = 0,
 \end{aligned}$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

Galerkin formulation

- In our numerical example, $U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}$ and $W_h = \text{span}\{\psi_j\}_{j=1}^{N_{bp}}$ are chosen to be the finite element spaces with the quadratic global basis functions $\{\phi_j\}_{j=1}^{N_b}$ and linear global basis functions $\{\psi_j\}_{j=1}^{N_{bp}}$, which are defined in Chapter 2. They are called **Taylor-Hood finite elements**.
- Why do we choose the pairs of finite elements in this way?
- Stability of mixed finite elements: **inf-sup condition**.

$$\inf_{0 \neq q_h \in W_h} \sup_{0 \neq \mathbf{u}_h \in U_h \times U_h} \frac{b(\mathbf{u}_h, q_h)}{\|\nabla \mathbf{u}_h\|_0 \|q_h\|_0} > \beta,$$

where $\beta > 0$ is a constant independent of mesh size h .

- See other course materials and references for the theory and more examples of stable mixed finite elements for unsteady Navier-Stokes equation.

Galerkin formulation

- In the scalar format, the Galerkin formulation is to find $u_{1h} \in H^1(0, T; U_h)$, $u_{2h} \in H^1(0, T; U_h)$, and $p_h \in L^2(0, T; W_h)$ such that

$$\begin{aligned}
 & \int_{\Omega} \frac{\partial u_{1h}}{\partial t} v_{1h} \, dx dy + \int_{\Omega} \frac{\partial u_{2h}}{\partial t} v_{2h} \, dx dy \\
 & + \int_{\Omega} \left(u_{1h} \frac{\partial u_{1h}}{\partial x} v_{1h} + u_{2h} \frac{\partial u_{1h}}{\partial y} v_{1h} + u_{1h} \frac{\partial u_{2h}}{\partial x} v_{2h} + u_{2h} \frac{\partial u_{2h}}{\partial y} v_{2h} \right) \, dx dy \\
 & + \int_{\Omega} \nu \left(2 \frac{\partial u_{1h}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}}{\partial y} \frac{\partial v_{1h}}{\partial y} \right. \\
 & \left. + \frac{\partial u_{1h}}{\partial y} \frac{\partial v_{2h}}{\partial x} + \frac{\partial u_{2h}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}}{\partial x} \frac{\partial v_{2h}}{\partial x} \right) \, dx dy \\
 & - \int_{\Omega} \left(p_h \frac{\partial v_{1h}}{\partial x} + p_h \frac{\partial v_{2h}}{\partial y} \right) \, dx dy = \int_{\Omega} (f_1 v_{1h} + f_2 v_{2h}) \, dx dy. \\
 & - \int_{\Omega} \left(\frac{\partial u_{1h}}{\partial x} q_h + \frac{\partial u_{2h}}{\partial y} q_h \right) \, dx dy = 0.
 \end{aligned}$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

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Full discretization

- Assume that we have a uniform partition of $[0, T]$ into M_m elements with mesh size Δt .
- The mesh nodes are $t_m = m\Delta t$, $m = 0, 1, \dots, M_m$.
- Let \mathbf{u}_h^0 and p_h^0 denote the given initial condition at t_0 .
- Let \mathbf{u}_h^m and p_h^m denote the numerical solution at t_m .
- For a simple illustration, we consider the full discretization with backward Euler scheme (without considering the Dirichlet boundary condition, which will be handled later): for $m = 0, \dots, M_m - 1$, find $\mathbf{u}_h^{m+1} \in [U_h]^2$ and $p_h^{m+1} \in W_h$ such that

$$\begin{aligned} & \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{\Delta t}, \mathbf{v} \right) + c(\mathbf{u}_h^{m+1}, \mathbf{u}_h^{m+1}, \mathbf{v}_h) + a(\mathbf{u}_h^{m+1}, \mathbf{v}_h) \\ & + b(\mathbf{v}_h, p_h^{m+1}) = (\mathbf{f}(t_{m+1}), \mathbf{v}_h), \\ & b(\mathbf{u}_h^{m+1}, q_h) = 0, \end{aligned}$$

Full discretization

- That is, for $m = 0, \dots, M_m - 1$, find $\mathbf{u}_h^{m+1} \in [U_h]^2$ and $p_h^{m+1} \in W_h$ such that

$$\begin{aligned}
 & \int_{\Omega} \frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{\Delta t} \cdot \mathbf{v}_h \, dx dy + \int_{\Omega} (\mathbf{u}_h^{m+1} \cdot \nabla) \mathbf{u}_h^{m+1} \cdot \mathbf{v}_h \, dx dy \\
 & + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}_h^{m+1}) : \mathbb{D}(\mathbf{v}_h) \, dx dy - \int_{\Omega} p_h^{m+1} (\nabla \cdot \mathbf{v}_h) \, dx dy \\
 & = \int_{\Omega} \mathbf{f}(t_{m+1}) \cdot \mathbf{v}_h \, dx dy, \\
 & - \int_{\Omega} (\nabla \cdot \mathbf{u}_h^{m+1}) q_h \, dx dy = 0,
 \end{aligned}$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

Full discretization

- For $m = 0, \dots, M_m - 1$, find $u_{1h}^{m+1}, u_{2h}^{m+1} \in U_h$ and $p_h^{m+1} \in W_h$ such that

$$\begin{aligned}
 & \int_{\Omega} \frac{u_{1h}^{m+1} - u_{1h}^m}{\Delta t} v_{1h} \, dx dy + \int_{\Omega} \frac{u_{2h}^{m+1} - u_{2h}^m}{\Delta t} v_{2h} \, dx dy + \int_{\Omega} \left(u_{1h}^{m+1} \frac{\partial u_{1h}^{m+1}}{\partial x} v_{1h} \right. \\
 & \left. + u_{2h}^{m+1} \frac{\partial u_{1h}^{m+1}}{\partial y} v_{1h} + u_{1h}^{m+1} \frac{\partial u_{2h}^{m+1}}{\partial x} v_{2h} + u_{2h}^{m+1} \frac{\partial u_{2h}^{m+1}}{\partial y} v_{2h} \right) dx dy \\
 & + \int_{\Omega} \nu \left(2 \frac{\partial u_{1h}^{m+1}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}^{m+1}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}^{m+1}}{\partial y} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{1h}^{m+1}}{\partial y} \frac{\partial v_{2h}}{\partial x} \right. \\
 & \left. + \frac{\partial u_{2h}^{m+1}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}^{m+1}}{\partial x} \frac{\partial v_{2h}}{\partial x} \right) dx dy - \int_{\Omega} \left(p_h^{m+1} \frac{\partial v_{1h}}{\partial x} + p_h^{m+1} \frac{\partial v_{2h}}{\partial y} \right) dx dy \\
 & = \int_{\Omega} f_1(t_{m+1}) v_{1h} \, dx dy + \int_{\Omega} f_2(t_{m+1}) v_{2h} \, dx dy, \\
 & - \int_{\Omega} \left(\frac{\partial u_{1h}^{m+1}}{\partial x} q_h + \frac{\partial u_{2h}^{m+1}}{\partial y} q_h \right) dx dy = 0,
 \end{aligned}$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

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- 2 Semi-discretization
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- 4 Newton's iteration**
- 5 Matrix formulation
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Newton's iteration

- How to handle the nonlinear terms in the full discretization?
- At each time iteration step of the full discretization, we have a steady nonlinear problem, which is similar to the steady Navier-Stokes equation.
- **Newton's iteration at each time iteration step!**
- Given the initial condition \mathbf{u}_h^0 and p_h^0 at the initial time

Newton's iteration at each time iteration step

At the $(m + 1)^{th}$ step ($m = 0, \dots, M_m - 1$) of the time iteration, we consider the following Newton's iteration:

- Initial guess: $\mathbf{u}_h^{m+1,(0)}$ and $p_h^{m+1,(0)}$. Usually they can be the solutions \mathbf{u}_h^m and p_h^m of the previous time iteration step.
- Newton's iteration for full discretization: for $l = 1, 2, \dots, L$, find $\mathbf{u}_h^{m+1,(l)} \in U_h \times U_h$ and $p_h^{m+1,(l)} \in W_h$ such that

$$\begin{aligned} & \left(\frac{\mathbf{u}_h^{m+1,(l)} - \mathbf{u}_h^m}{\Delta t}, \mathbf{v} \right) + c(\mathbf{u}_h^{m+1,(l)}, \mathbf{u}_h^{m+1,(l-1)}, \mathbf{v}_h) \\ & + c(\mathbf{u}_h^{m+1,(l-1)}, \mathbf{u}_h^{m+1,(l)}, \mathbf{v}_h) + a(\mathbf{u}_h^{m+1,(l)}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^{m+1,(l)}) \\ & = (\mathbf{f}(t_{m+1}), \mathbf{v}_h) + c(\mathbf{u}_h^{m+1,(l-1)}, \mathbf{u}_h^{m+1,(l-1)}, \mathbf{v}_h), \\ & b(\mathbf{u}_h^{m+1,(l)}, q_h) = 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

- Let \mathbf{u}_h^{m+1} be the final $\mathbf{u}_h^{m+1,(l)}$ from the above iteration.

Newton's iteration at each time iteration step

- Initial guess: $\mathbf{u}_h^{m+1,(0)}$ and $p_h^{m+1,(0)}$. Usually they can be the solutions \mathbf{u}_h^m and p_h^m of the previous time iteration step.
- Newton's iteration for full discretization in the vector format: for $l = 1, 2, \dots, L$, find $\mathbf{u}_h^{m+1,(l)} \in U_h \times U_h$ and $p_h^{m+1,(l)} \in W_h$ such that

$$\begin{aligned} & \int_{\Omega} \frac{\mathbf{u}_h^{m+1,(l)} - \mathbf{u}_h^m}{\Delta t} \cdot \mathbf{v}_h \, dx dy + \int_{\Omega} (\mathbf{u}_h^{m+1,(l)} \cdot \nabla) \mathbf{u}_h^{m+1,(l-1)} \cdot \mathbf{v}_h \, dx dy \\ & + \int_{\Omega} (\mathbf{u}_h^{m+1,(l-1)} \cdot \nabla) \mathbf{u}_h^{m+1,(l)} \cdot \mathbf{v}_h \, dx dy \\ & + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}_h^{m+1,(l)}) : \mathbb{D}(\mathbf{v}_h) \, dx dy - \int_{\Omega} p_h^{m+1,(l)} (\nabla \cdot \mathbf{v}_h) \, dx dy \\ & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h \, dx dy + \int_{\Omega} (\mathbf{u}_h^{m+1,(l-1)} \cdot \nabla) \mathbf{u}_h^{m+1,(l-1)} \cdot \mathbf{v}_h \, dx dy, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}_h^{m+1,(l)}) q_h \, dx dy = 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

- Let \mathbf{u}_h^{m+1} be the final $\mathbf{u}_h^{m+1,(l)}$ from the above iteration.

Newton's iteration at each time iteration step

- Initial guess: $u_{1h}^{m+1,(0)}$, $u_{2h}^{m+1,(0)}$, and $p_h^{m+1,(0)}$. Usually they can be the solutions u_{1h}^m , u_{2h}^m , and p_h^m of the previous time iteration step.
- Newton's iteration for full discretization in the scalar format:
for $l = 1, 2, \dots, L$, find $u_{1h}^{m+1,(l)} \in U_h$, $u_{2h}^{m+1,(l)} \in U_h$, and $p_h^{m+1,(l)} \in W_h$ such that

Newton's iteration at each time iteration step

$$\begin{aligned}
& \int_{\Omega} \frac{u_{1h}^{m+1,(l)} - u_{1h}^m}{\Delta t} v_{1h} \, dx dy + \int_{\Omega} \frac{u_{2h}^{m+1,(l)} - u_{2h}^m}{\Delta t} v_{2h} \, dx dy + \int_{\Omega} \left(u_{1h}^{m+1,(l)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} v_{1h} \right. \\
& + u_{2h}^{m+1,(l)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} v_{1h} + u_{1h}^{m+1,(l)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} v_{2h} + u_{2h}^{m+1,(l)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} v_{2h} \left. \right) dx dy \\
& + \int_{\Omega} \left(u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l)}}{\partial x} v_{1h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} v_{1h} + u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} v_{2h} \right. \\
& \left. + u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} v_{2h} \right) dx dy \\
& + \int_{\Omega} \nu \left(2 \frac{\partial u_{1h}^{m+1,(l)}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} \frac{\partial v_{2h}}{\partial x} \right. \\
& \left. + \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} \frac{\partial v_{2h}}{\partial x} \right) dx dy - \int_{\Omega} \left(p_h^{m+1,(l)} \frac{\partial v_{1h}}{\partial x} + p_h^{m+1,(l)} \frac{\partial v_{2h}}{\partial y} \right) dx dy \\
& = \int_{\Omega} (f_1 v_{1h} + f_2 v_{2h}) \, dx dy + \int_{\Omega} \left(u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} v_{1h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} v_{1h} \right. \\
& \left. + u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} v_{2h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} v_{2h} \right) dx dy,
\end{aligned}$$

Newton's iteration at each time iteration step

- Continued formulation:

$$- \int_{\Omega} \left(\frac{\partial u_{1h}^{m+1,(l)}}{\partial x} q_h + \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} q_h \right) dx dy = 0.$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

- Let u_{1h}^{m+1} and u_{2h}^{m+1} be the final $u_{1h}^{m+1,(l)}$ and $u_{2h}^{m+1,(l)}$ from the above iteration.

Outline

- 1 Weak formulation
- 2 Semi-discretization
- 3 Full discretization
- 4 Newton's iteration
- 5 Matrix formulation**
- 6 FE method
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Matrix formulation

- Since $u_{1h}^{m+1,(l)}$, $u_{2h}^{m+1,(l)}$, u_{1h}^m , $u_{2h}^m \in U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}$ and $p_h^{m+1,(l)}$, $p_h^m \in W_h = \text{span}\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{m+1,(l)} = \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \phi_j, \quad u_{1h}^m = \sum_{j=1}^{N_b} u_{1j}^m \phi_j,$$

$$u_{2h}^{m+1,(l)} = \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \phi_j, \quad u_{2h}^m = \sum_{j=1}^{N_b} u_{2j}^m \phi_j$$

$$p_h^{m+1,(l)} = \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \psi_j, \quad p_h^m = \sum_{j=1}^{N_{bp}} p_j^m \psi_j,$$

for some coefficients $u_{1j}^{m+1,(l)}$, $u_{2j}^{m+1,(l)}$, u_{1j}^m , u_{2j}^m
 ($j = 1, \dots, N_b$), and $p_j^{m+1,(l)}$, p_j^m , ($j = 1, \dots, N_{bp}$).

Matrix formulation

- If we can set up a linear algebraic system for $u_{1j}^{m+1,(l)}$, $u_{2j}^{m+1,(l)}$ ($j = 1, \dots, N_b$), and $p_j^{m+1,(l)}$ ($j = 1, \dots, N_{bp}$), then we can solve it to obtain the finite element solution $\mathbf{u}_h^{m+1,(l)} = (u_{1h}^{m+1,(l)}, u_{2h}^{m+1,(l)})^t$ and $p_h^{m+1,(l)}$ at the step l ($l = 1, 2, \dots, L$) of Newton's iteration.
- For the first equation at the step l ($l = 1, 2, \dots, L$) of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ ($i = 1, \dots, N_b$) and $\mathbf{v}_h = (0, \phi_i)^t$ ($i = 1, \dots, N_b$). That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ ($i = 1, \dots, N_b$) and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ ($i = 1, \dots, N_b$).
- For the second equation at the step l ($l = 1, 2, \dots, L$) of Newton's iteration, we choose $q_h = \psi_i$ ($i = 1, \dots, N_{bp}$).

Matrix formulation

- Set $\mathbf{v}_h = (\phi_i, 0)^t$, i.e., $v_{1h} = \phi_i$ and $v_{2h} = 0$ ($i = 1, \dots, N_b$), in the first equation at the step l ($l = 1, 2, \dots, L$) of Newton's iteration. Then

$$\begin{aligned}
 & \frac{1}{\Delta t} \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy - \frac{1}{\Delta t} \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{1j}^m \phi_j \right) \phi_i \, dx dy \\
 & + \int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy + \int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy \\
 & + \int_{\Omega} u_{1h}^{m+1,(l-1)} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \phi_i \, dx dy + \int_{\Omega} u_{2h}^{m+1,(l-1)} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \phi_i \, dx dy \\
 & + 2 \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \frac{\partial \phi_i}{\partial x} \, dx dy + \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \frac{\partial \phi_i}{\partial y} \, dx dy \\
 & + \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \frac{\partial \phi_i}{\partial y} \, dx dy - \int_{\Omega} \left(\sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \psi_j \right) \frac{\partial \phi_i}{\partial x} \, dx dy \\
 & = \int_{\Omega} f_1 \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy.
 \end{aligned}$$

Matrix formulation

- Set $\mathbf{v}_h = (0, \phi_i)^t$, i.e., $v_{1h} = 0$ and $v_{2h} = \phi_i$ ($i = 1, \dots, N_b$), in the first equation of the Galerkin formulation at the step l ($l = 1, 2, \dots, L$) of Newton's iteration. Then

$$\begin{aligned}
 & \frac{1}{\Delta t} \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy - \frac{1}{\Delta t} \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{2j}^m \phi_j \right) \phi_i \, dx dy \\
 & + \int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy + \int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \phi_j \right) \phi_i \, dx dy \\
 & + \int_{\Omega} u_{1h}^{m+1,(l-1)} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \phi_i \, dx dy + \int_{\Omega} u_{2h}^{m+1,(l-1)} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \phi_i \, dx dy \\
 & + 2 \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \frac{\partial \phi_i}{\partial y} \, dx dy + \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \frac{\partial \phi_i}{\partial x} \, dx dy \\
 & + \int_{\Omega} \nu \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \frac{\partial \phi_i}{\partial x} \, dx dy - \int_{\Omega} \left(\sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \psi_j \right) \frac{\partial \phi_i}{\partial y} \, dx dy \\
 & = \int_{\Omega} f_2 \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy.
 \end{aligned}$$

Matrix formulation

- Set $q_h = \psi_i$ ($i = 1, \dots, N_{bp}$) in the second equation of the Galerkin formulation at the step l ($l = 1, 2, \dots, L$) of Newton's iteration. Then

$$- \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \psi_i \, dx dy - \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \psi_i \, dx dy = 0.$$

Matrix formulation

- Simplify the above three sets of equations, we obtain

$$\begin{aligned}
 & \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \left(\frac{1}{\Delta t} \int_{\Omega} \phi_j \phi_i \, dx dy + 2 \int_{\Omega} \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx dy + \int_{\Omega} \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dx dy \right. \\
 & + \int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_j \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dx dy \\
 & \left. + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dx dy \right) + \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \left(\int_{\Omega} \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial y} \, dx dy \right. \\
 & \left. + \int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_j \phi_i \, dx dy \right) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \left(- \int_{\Omega} \psi_j \frac{\partial \phi_i}{\partial x} \, dx dy \right) \\
 & = \int_{\Omega} f_1 \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy \\
 & + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy + \sum_{j=1}^{N_b} u_{1j}^m \left(\frac{1}{\Delta t} \int_{\Omega} \phi_j \phi_i \, dx dy \right),
 \end{aligned}$$

Matrix formulation

- Continued formulation:

$$\begin{aligned}
 & \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \left(\int_{\Omega} \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \, dx dy + \int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_j \phi_i \, dx dy \right) \\
 & + \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \left(\frac{1}{\Delta t} \int_{\Omega} \phi_j \phi_i \, dx dy + 2 \int_{\Omega} \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dx dy \right. \\
 & + \int_{\Omega} \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx dy + \int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_j \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dx dy \\
 & \left. + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dx dy \right) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \left(- \int_{\Omega} \psi_j \frac{\partial \phi_i}{\partial y} \, dx dy \right) \\
 & = \int_{\Omega} f_2 \phi_i \, dx dy + \int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy \\
 & + \int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy + \sum_{j=1}^{N_b} u_{2j}^m \left(\frac{1}{\Delta t} \int_{\Omega} \phi_j \phi_i \, dx dy \right),
 \end{aligned}$$

Matrix formulation

- Continued formulation:

$$\begin{aligned}
 & \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \left(- \int_{\Omega} \frac{\partial \phi_j}{\partial x} \psi_i \, dx dy \right) \\
 & + \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \left(- \int_{\Omega} \frac{\partial \phi_j}{\partial y} \psi_i \, dx dy \right) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} * 0 \\
 = & 0.
 \end{aligned}$$

Matrix formulation

- Define

$$\begin{aligned}
 A_1 &= \left[\int_{\Omega} \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} dx dy \right]_{i,j=1}^{N_b}, & A_2 &= \left[\int_{\Omega} \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} dx dy \right]_{i,j=1}^{N_b}, \\
 A_3 &= \left[\int_{\Omega} \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial y} dx dy \right]_{i,j=1}^{N_b}, & A_4 &= \left[\int_{\Omega} \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} dx dy \right]_{i,j=1}^{N_b}, \\
 A_5 &= \left[\int_{\Omega} -\psi_j \frac{\partial \phi_i}{\partial x} dx dy \right]_{i=1,j=1}^{N_b, N_{bp}}, & A_6 &= \left[\int_{\Omega} -\psi_j \frac{\partial \phi_i}{\partial y} dx dy \right]_{i=1,j=1}^{N_b, N_{bp}}, \\
 A_7 &= \left[\int_{\Omega} -\frac{\partial \phi_j}{\partial x} \psi_i dx dy \right]_{i=1,j=1}^{N_{bp}, N_b}, & A_8 &= \left[\int_{\Omega} -\frac{\partial \phi_j}{\partial y} \psi_i dx dy \right]_{i=1,j=1}^{N_{bp}, N_b}.
 \end{aligned}$$

- Define a zero matrix $\mathbb{O}_1 = [0]_{i=1,j=1}^{N_{bp}, N_{bp}}$ whose size is $N_{bp} \times N_{bp}$. Then

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_4 & 2A_2 + A_1 & A_6 \\ A_7 & A_8 & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Each matrix above can be obtained by Algorithm I-3 in Chapter 3.
- It is not difficult to verify (an independent study project topic) that

$$A_4 = A_3^t, \quad A_7 = A_5^t, \quad A_8 = A_6^t.$$

- Hence the matrix A is actually symmetric:

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

Another format of full discretization

- Define the basic mass matrix

$$M_e = [m_{ij}]_{i,j=1}^{N_b} = \left[\int_{\Omega} \phi_j \phi_i \, dx dy \right]_{i,j=1}^{N_b}.$$

- The mass matrix M_e can be obtained by Algorithm I-3 in Chapter 3, with $r = s = p = q = 0$ and $c = 1$.
- Define zero matrices $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b, N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b, N_b}$. Then define the block mass matrix

$$M = \begin{pmatrix} M_e & \mathbb{O}_3 & \mathbb{O}_2 \\ \mathbb{O}_3 & M_e & \mathbb{O}_2 \\ \mathbb{O}_2^t & \mathbb{O}_2^t & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Define

$$AN_1 = \left[\int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_j \phi_i \, dx dy \right]_{i,j=1}^{N_b}, \quad AN_2 = \left[\int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dx dy \right]_{i,j=1}^{N_b}$$

$$AN_3 = \left[\int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dx dy \right]_{i,j=1}^{N_b}, \quad AN_4 = \left[\int_{\Omega} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_j \phi_i \, dx dy \right]_{i,j=1}^{N_b}$$

$$AN_5 = \left[\int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_j \phi_i \, dx dy \right]_{i,j=1}^{N_b}, \quad AN_6 = \left[\int_{\Omega} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_j \phi_i \, dx dy \right]_{i,j=1}^{N_b}$$

- Then

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_2 \\ \mathbb{O}_2^t & \mathbb{O}_2^t & \mathbb{O}_1 \end{pmatrix}$$

- Each matrix above can be obtained by Algorithm VIII in Chapter 8.

Matrix formulation

- Define the load vector

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_{\Omega} f_1 \phi_i dx dy \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_{\Omega} f_2 \phi_i dx dy \right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each of \vec{b}_1 and \vec{b}_2 can be obtained by Algorithm II-3 in Chapter 3.

Matrix formulation

- Define the vector

$$\vec{bN} = \begin{pmatrix} \vec{bN}_1 + \vec{bN}_2 \\ \vec{bN}_3 + \vec{bN}_4 \\ \vec{0} \end{pmatrix}$$

where $\vec{0} = [0]_{i=1}^{N_{bp}}$ and

$$\vec{bN}_1 = \left[\int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy \right]_{i=1}^{N_b},$$

$$\vec{bN}_2 = \left[\int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy \right]_{i=1}^{N_b},$$

$$\vec{bN}_3 = \left[\int_{\Omega} u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_i \, dx dy \right]_{i=1}^{N_b},$$

$$\vec{bN}_4 = \left[\int_{\Omega} u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_i \, dx dy \right]_{i=1}^{N_b}.$$

Matrix formulation

- Each vector above can be obtained by Algorithm IX in Chapter 8.
- Define the known vector from the previous time iteration step:

$$\vec{X}^m = \begin{pmatrix} \vec{X}_1^m \\ \vec{X}_2^m \\ \vec{X}_3^m \end{pmatrix}$$

where

$$\vec{X}_1^m = [u_{1j}^m]_{j=1}^{N_b},$$

$$\vec{X}_2^m = [u_{2j}^m]_{j=1}^{N_b},$$

$$\vec{X}_3^m = [p_j^m]_{j=1}^{N_{bp}}.$$

Matrix formulation

- Define the unknown vector

$$\vec{X}^{m+1,(l)} = \begin{pmatrix} \vec{X}_1^{m+1,(l)} \\ \vec{X}_2^{m+1,(l)} \\ \vec{X}_3^{m+1,(l)} \end{pmatrix}$$

where

$$\vec{X}_1^{m+1,(l)} = \left[u_{1j}^{m+1,(l)} \right]_{j=1}^{N_b},$$

$$\vec{X}_2^{m+1,(l)} = \left[u_{2j}^{m+1,(l)} \right]_{j=1}^{N_b},$$

$$\vec{X}_3^{m+1,(l)} = \left[p_j^{m+1,(l)} \right]_{j=1}^{N_{bp}}.$$

Matrix formulation

- Define

$$A^{m+1,(l)} = \frac{M}{\Delta t} + A + AN, \quad \vec{b}^{m+1,(l)} = \vec{b} + \frac{M}{\Delta t} \vec{X}^m + \vec{b}N.$$

- For step l ($l = 1, 2, \dots, L$) of the Newton's iteration at the $(m+1)^{th}$ step of the time iteration, we obtain the linear algebraic system

$$A^{m+1,(l)} \vec{X}^{m+1,(l)} = \vec{b}^{m+1,(l)}.$$

- Let X^{m+1} be the final $\vec{X}^{m+1,(l)}$ from the above Newton's iteration at the $(m+1)^{th}$ step of the time iteration.

Outline

- 1 Weak formulation
- 2 Semi-discretization
- 3 Full discretization
- 4 Newton's iteration
- 5 Matrix formulation
- 6 FE method**
- 7 More Discussion

Assembly of a time-independent matrix

Recall Algorithm I-3 from Chapter 3:

- Initialize the matrix: $A = \text{sparse}(N_b, N_b)$;
- Compute the integrals and assemble them into A :

FOR $n = 1, \dots, N$:

 FOR $\alpha = 1, \dots, N_{Ib}$:

 FOR $\beta = 1, \dots, N_{Ib}$:

 Compute $r = \int_{E_n} c \frac{\partial^{r+s} \psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} dx dy$;

 Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

 END

 END

END

Assembly of the time-independent stiffness matrix

- Call **Algorithm I-3** with $r = 1, s = 0, p = 1, q = 0, c = \nu$, basis type of \mathbf{u} for trial function, and basis type of \mathbf{u} for test function, to obtain A_1 .
- Call **Algorithm I-3** with $r = 0, s = 1, p = 0, q = 1, c = \nu$, basis type of \mathbf{u} for trial function, and basis type of \mathbf{u} for test function, to obtain A_2 .
- Call **Algorithm I-3** with $r = 1, s = 0, p = 0, q = 1, c = \nu$, basis type of \mathbf{u} for trial function, and basis type of \mathbf{u} for test function, to obtain A_3 .
- Call **Algorithm I-3** with $r = 0, s = 0, p = 1, q = 0, c = -1$, basis type of p for trial function, and basis type of \mathbf{u} for test function, to obtain A_5 .
- Call **Algorithm I-3** with $r = 0, s = 0, p = 0, q = 1, c = -1$, basis type of p for trial function, and basis type of \mathbf{u} for test function, to obtain A_6 .
- Generate a zero matrix \mathbb{O} whose size is $N_{bp} \times N_{bp}$.
- Then the stiffness matrix

$$A = [A_1 + 2A_2 \quad A_3 \quad A_5; A_3^t \quad 2A_2 + A_1 \quad A_6; A_5^t \quad A_6^t \quad \mathbb{O}].$$

Assembly of the mass matrix

- Call **Algorithm I-3** with $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c = 1$, basis type of \mathbf{u} for trial function, and basis type of \mathbf{u} for test function, to obtain the basic mass matrix M_e .
- Generate three zero matrices \mathbb{O}_1 , \mathbb{O}_2 , and \mathbb{O}_3 whose sizes are $N_{bp} \times N_{bp}$, $N_b \times N_{bp}$, and $N_b \times N_b$, respectively.
- Then the block mass matrix

$$M = [M_e \quad \mathbb{O}_3 \quad \mathbb{O}_2; \mathbb{O}_3 \quad M_e \quad \mathbb{O}_2; \mathbb{O}_2^t \quad \mathbb{O}_2^t \quad \mathbb{O}_1].$$

Assembly of a time-independent vector

Recall Algorithm II-3 from Chapter 3:

- Initialize the matrix: $b = \text{sparse}(N_b, 1)$;
- Compute the integrals and assemble them into b :

FOR $n = 1, \dots, N$:

FOR $\beta = 1, \dots, N_{lb}$:

 Compute $r = \int_{E_n} f \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} dx dy$;

$b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

END

END

Assembly of a time-dependent vector

Recall Algorithm II-5 from Chapter 4:

- Specify a value for the time t based on the input time;
- Initialize the vector: $b = \text{sparse}(N_b, 1)$;
- Compute the integrals and assemble them into b :

FOR $n = 1, \dots, N$:

FOR $\beta = 1, \dots, N_{Ib}$:

Compute $r = \int_{E_n} f(t) \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} dx dy$;

$b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

END

END

Assembly of the load vector

- Call **Algorithm II-5** with $p = q = 0$ and $f = f_1$ to obtain $b_1(t)$.
- Call **Algorithm II-5** with $p = q = 0$ and $f = f_2$ to obtain $b_2(t)$.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$.
- Then the load vector $\vec{b} = [b_1(t); b_2(t); \vec{0}]$.
- If f_1 and f_2 do not depend on t , then this part is exactly the same as the assembly of the load vector with Algorithm II-3 in Chapter 7.

Assembly of a matrix for an integral with a finite element coefficient function

Recall Algorithm VIII from Chapter 8:

- Initialize the matrix: $A = \text{sparse}(N_b, N_b)$;
- Compute the integrals and assemble them into A :

FOR $n = 1, \dots, N$:

FOR $\alpha = 1, \dots, N_{lb}$:

FOR $\beta = 1, \dots, N_{lb}$:

Compute $r = \int_{E_n} \frac{\partial^{d+e} c_h}{\partial x^d \partial y^e} \frac{\partial^{r+s} \psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} dx dy$;

Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

END

END

END

Assembly of a matrix for an integral with a finite element coefficient function

- Call **Algorithm VIII** with $d = 1, e = 0, r = 0, s = 0, p = 0, q = 0, c_h = u_{1h}^{(j-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_1 .
- Call **Algorithm VIII** with $d = 0, e = 0, r = 1, s = 0, p = 0, q = 0, c_h = u_{1h}^{(j-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_2 .
- Call **Algorithm VIII** with $d = 0, e = 0, r = 0, s = 1, p = 0, q = 0, c_h = u_{2h}^{(j-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_3 .
- Call **Algorithm VIII** with $d = 0, e = 1, r = 0, s = 0, p = 0, q = 0, c_h = u_{1h}^{(j-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_4 .

Assembly of a matrix for an integral with a finite element coefficient function

- Call **Algorithm VIII** with $d = 1, e = 0, r = 0, s = 0, p = 0, q = 0, c_h = u_{2h}^{(l-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_5 .
- Call **Algorithm VIII** with $d = 0, e = 1, r = 0, s = 0, p = 0, q = 0, c_h = u_{2h}^{(l-1)}$, basis type of \mathbf{u} for both trial and test functions, to obtain AN_6 .
- Generate a zero matrix $\mathbb{O}_1 = [0]_{i,j=1}^{N_{bp}}$, $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b, N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b, N_{bp}}$.
- Then the stiffness matrix

$$A = [AN_1 + AN_2 + AN_3 \quad AN_4 \quad \mathbb{O}_2; AN_5 \quad AN_6 + AN_2 + AN_3 \quad \mathbb{O}_3; \mathbb{O}_2^t \quad \mathbb{O}_3^t \quad \mathbb{O}_1].$$

Assembly of the vector for an integral with two finite element coefficient functions

Recall Algorithm IX from Chapter 8:

- Initialize the vector: $b = \text{sparse}(N_b, 1)$;
- Compute the integrals and assemble them into b :

FOR $n = 1, \dots, N$:

FOR $\beta = 1, \dots, N_{Ib}$:

Compute $r = \int_{E_n} \frac{\partial^{d+e} f_{1h}}{\partial x^d \partial y^e} \frac{\partial^{r+s} f_{2h}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} dx dy$;

$b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

END

END

Assembly of the vector for an integral with two finite element coefficient functions

- Call **Algorithm IX** with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$ and $f_{h1} = u_{1h}^{(l-1)}$, $f_{h2} = u_{1h}^{(l-1)}$ to obtain bN_1 .
- Call **Algorithm IX** with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$ and $f_{h1} = u_{2h}^{(l-1)}$, $f_{h2} = u_{1h}^{(l-1)}$ to obtain bN_2 .
- Call **Algorithm IX** with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$ and $f_{h1} = u_{1h}^{(l-1)}$, $f_{h2} = u_{2h}^{(l-1)}$ to obtain bN_3 .
- Call **Algorithm IX** with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$ and $f_{h1} = u_{2h}^{(l-1)}$, $f_{h2} = u_{2h}^{(l-1)}$ to obtain bN_4 .
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$
- Then the load vector $\vec{bN} = [bN_1 + bN_2; bN_3 + bN_4; \vec{0}]$.

Time-dependent Dirichlet boundary condition

Recall Algorithm III-4 from Chapter 9:

- Specify a value for the time t based on the input time;
- Deal with the Dirichlet boundary conditions:

FOR $k = 1, \dots, nbn$:

 If *boundarynodes*(1, k) shows Dirichlet condition, then

$i = \text{boundarynodes}(2, k)$;

$\tilde{A}(i, :) = 0$;

$\tilde{A}(i, i) = 1$;

$\tilde{b}(i) = g_1(P_b(:, i), t)$;

$\tilde{A}(N_b + i, :) = 0$;

$\tilde{A}(N_b + i, N_b + i) = 1$;

$\tilde{b}(N_b + i) = g_2(P_b(:, i), t)$;

 ENDIF

END

Main pseudo code

Algorithm B:

- Generate the mesh information matrices P and T .
- Assemble the mass matrix M and stiffness matrix A by using **Algorithm I-3**.
- Generate the initial vector \vec{X}^0 .
- Iterate in time: *FOR* $m = 0, \dots, M_m - 1$
 - $t_{m+1} = (m + 1)\Delta t$;
 - Assemble the load vector \vec{b} by using **Algorithm II-5**.
 - Newton iteration: *FOR* $l = 1, 2, \dots, L$
 - Assemble the matrix AN by using **Algorithm VIII**.
 - Assemble the vector \vec{bN} by using **Algorithm IX**.
 - $A^{m+1,(l)} = \frac{M}{\Delta t} + A + AN$ and $\vec{b}^{m+1,(l)} = \vec{b} + \frac{M}{\Delta t}\vec{X}^m + \vec{bN}$
 - Treat Dirichlet boundary for $A^{m+1,(l)}$ and $\vec{b}^{m+1,(l)}$ by **Algorithm III-4**.
 - Solve $A^{m+1,(l)}\vec{X}^{m+1,(l)} = \vec{b}^{m+1,(l)}$ for \vec{X} .

END

- Let X^{m+1} be the final $\vec{X}^{m+1,(l)}$ from the above Newton's iteration.

END

Numerical example

- Example 1: On the domain $\Omega = [0, 1] \times [-0.25, 0]$, consider the time-dependent Navier-Stokes equation

$$\begin{aligned}\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) &= \mathbf{f} && \text{in } \Omega \times [0, 1], \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times [0, 1].\end{aligned}$$

Numerical example

Independent study topic:

- (1) Following the traditional way, which was used to set up the numerical examples in the previous chapters, determine the source term \mathbf{f} , initial condition, Dirichlet boundary conditions, and fixed value of p at $(0,0)$ such that the analytic solutions of this problem are

$$u_1 = (x^2y^2 + e^{-y})\cos(2\pi t),$$

$$u_2 = \left[-\frac{2}{3}xy^3 + 2 - \pi \sin(\pi x) \right] \cos(2\pi t),$$

$$p = -[2 - \pi \sin(\pi x)] \cos(2\pi y) \cos(2\pi t).$$

- (2) Choose $h = 1/8, 1/16, 1/32$ and $\Delta t = 8h^3$. Use the Taylor-Hood finite elements with backward Euler scheme to solve this equation and provide the numerical errors of \mathbf{u} and p in L^2, L^∞ , and H^1 norms.

Outline

- 1 Weak formulation
- 2 Semi-discretization
- 3 Full discretization
- 4 Newton's iteration
- 5 Matrix formulation
- 6 FE method
- 7 More Discussion**

Mixed boundary conditions

- The treatment of the stress/Robin boundary conditions is similar to that of Chapter 7.
- If the functions in the stress/Robin boundary conditions are independent of time, then the same subroutines from Chapter 7 can be used before the time iteration starts.
- If the functions in the stress/Robin boundary conditions depend on time, then the same algorithms as those in Chapter 7 can be used at each time iteration step. But the time needs to be specified in these algorithms.

Mixed boundary conditions

- Consider

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \times [0, T],$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T],$$

$$\mathbb{T}(\mathbf{u}, p) \mathbf{n} = \mathbf{p} \quad \text{on } \Gamma_S \times [0, T],$$

$$\mathbb{T}(\mathbf{u}, p) \mathbf{n} + r \mathbf{u} = \mathbf{q} \quad \text{on } \Gamma_R \times [0, T],$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D \times [0, T],$$

$$\mathbf{u} = \mathbf{u}_0, \quad \text{at } t = 0 \text{ and in } \Omega.$$

where $\Gamma_S, \Gamma_R \subset \partial\Omega$ and $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$.

Mixed boundary conditions

- Recall

$$\begin{aligned} & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\ & - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0. \end{aligned}$$

- Since the solution on $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega / (\Gamma_S \cup \Gamma_R)$.

Mixed boundary conditions

- Hence, similar to the treatment of the mixed boundary condition in Chapter 7, the weak formulation is to find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned} & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy \\ & + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy + \int_{\Gamma_R} r \mathbf{u} \cdot \mathbf{v} \, ds \\ & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \, ds + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \, ds, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0. \end{aligned}$$

for any $\mathbf{v} \in [H_{0D}^1(\Omega)]^2$ and $q \in L^2(\Omega)$ where $H_{0D}^1(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D\}$.

- Code? Combine all of the subroutines for Dirichlet/Stress/Robin boundary conditions.

Mixed boundary conditions in normal/tangential directions

- Consider

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \times [0, T],$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T],$$

$$\mathbf{n}^t \mathbb{T}(\mathbf{u}, p) \mathbf{n} = p_n, \tau^t \mathbb{T}(\mathbf{u}, p) \mathbf{n} = p_\tau \quad \text{on } \Gamma_S \times [0, T],$$

$$\mathbf{n}^t \mathbb{T}(\mathbf{u}, p) \mathbf{n} + r \mathbf{n}^t \mathbf{u} = q_n, \tau^t \mathbb{T}(\mathbf{u}, p) \mathbf{n} + r \tau^t \mathbf{u} = q_\tau \quad \text{on } \Gamma_R \times [0, T],$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D \times [0, T],$$

$$\mathbf{u} = \mathbf{u}_0, \quad \text{at } t = 0 \text{ and in } \Omega.$$

where $\Gamma_S, \Gamma_R \subset \partial\Omega$, $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$, $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of $\partial\Omega$, and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Mixed boundary conditions in normal/tangential directions

- Recall

$$\begin{aligned} & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\ & - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy, \\ & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0. \end{aligned}$$

- Since the solution on $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega / (\Gamma_S \cup \Gamma_R)$.

Mixed boundary conditions in normal/tangential directions

- Similar to the derivation of mixed boundary conditions in normal/tangential directions in Chapter 7, we obtain

$$\begin{aligned}
 & \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds \\
 = & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds \\
 & + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \, ds \\
 = & \left[\int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_S} p_\tau(\boldsymbol{\tau}^t \mathbf{v}) \, ds \right] \\
 & + \left[\int_{\Gamma_R} q_n(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_R} q_\tau(\boldsymbol{\tau}^t \mathbf{v}) \, ds \right] \\
 & - \left[\int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u})(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_R} (r\boldsymbol{\tau}^t \mathbf{u})(\boldsymbol{\tau}^t \mathbf{v}) \, ds \right],
 \end{aligned}$$

Mixed boundary conditions in normal/tangential directions

- Hence, similar to the treatment of the mixed boundary conditions in normal/tangential directions in Chapter 7, the weak formulation is to find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned}
 & \int_{\Omega} \mathbf{u}_t \cdot \mathbf{v} \, dx dy + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, dx dy + \int_{\Omega} 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dx dy \\
 & - \int_{\Omega} p(\nabla \cdot \mathbf{v}) \, dx dy + \int_{\Gamma_R} (r \mathbf{n}^t \mathbf{u})(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_R} (r \boldsymbol{\tau}^t \mathbf{u})(\boldsymbol{\tau}^t \mathbf{v}) \, ds \\
 & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx dy + \int_{\Gamma_R} q_n(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_R} q_{\tau}(\boldsymbol{\tau}^t \mathbf{v}) \, ds \\
 & \quad + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_S} p_{\tau}(\boldsymbol{\tau}^t \mathbf{v}) \, ds, \\
 & - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dx dy = 0,
 \end{aligned}$$

for any $\mathbf{v} \in [H_{0D}^1(\Omega)]^2$ and $q \in L^2(\Omega)$.

- Code? Combine all of the subroutines for Dirichlet/Stress/Robin boundary conditions.